Non-Markovian evolution of photonic quantum states in atmospheric turbulence

2016 J. Opt. 18 055203

(http://iopscience.iop.org/2040-8986/18/5/055203)

IOP Publishing Journal of Optics
J. Opt. **18** (2016) 055203 (11pp) doi:10.1088/2040-8978/18/5/055203

Non-Markovian evolution of photonic quantum states in atmospheric turbulence

Filippus S Roux

CSIR National Laser Centre, PO Box 395, Pretoria 0001, South Africa
School of Physics, University of Witwatersrand, Johannesburg 2000, South Africa

E-mail: fsroux@csir.co.za

Received 5 January 2016, revised 10 February 2016
Accepted for publication 10 February 2016
Published 15 March 2016

Abstract

The evolution of the spatial degrees of freedom of a photon propagating through atmospheric turbulence is treated as a non-Markovian process. Here, we derive and solve the evolution equation for this process. The turbulent medium is modeled by a sequence of multiple phase screens for general turbulence conditions. The non-Markovian perspective leads to a second-order differential equation with respect to the propagation distance. The solution for this differential equation is obtained with the aid of a perturbative analysis, assuming the turbulence is relatively weak. We also provide another solution for more general turbulence strengths, but where we introduce a simplification to the differential equation.

Keywords: non-Markovian process, atmospheric turbulence, perturbative analysis, photonic quantum state

1. Introduction

The scintillation that a photonic quantum state experiences as it propagates through a turbulent atmosphere is a topic of considerable importance for free-space quantum communication. Apart from the laboratory experiments on the propagation of photonic quantum states through simulated turbulence [1–3], a number of experimental demonstrations of quantum key distribution through free-space has also been done [4–6]. The evolution of the quantum state in these scenarios can be considered using a single phase screen (SPS) model [7], provided that the scintillation remains weak. While the SPS model is used in most of the work that has been done in this field [1, 8–13], a more accurate multiple phase screen (MPS) approach has been proposed recently [14–17]. The MPS approach is based on the principle of infinitesimal propagation, which allows one to derive an equation for the evolution of the quantum state, called the infinitesimal propagation equation (IPE). The IPE is a first-order differential equation with respect to the propagation distance, which can be solved [17] to obtain an expression for the density matrix of the quantum state at arbitrary propagation distances and under arbitrary turbulence conditions.

However, all previous analyses of the evolution of photonic quantum states propagating through turbulence, including the IPE, employed a Markov approximation. In this approximation it is assumed that the medium is delta-correlated with itself along the propagation direction. For the derivation of the first-order differential equation of the IPE, one effectively assumes that the infinitesimal propagation step size is larger than the intrinsic scale, which in this case is the outer scale of the turbulence. To obtain the differential equation, one takes the limit where the step size goes to zero. On the other hand, the outer scale is assumed to go to infinity, allowing one to use the Kolmogorov turbulence model [18]. This seems to be a clear contradiction without a suitable justification. Between the inner and outer scale the Kolmogorov spectrum of turbulence exhibits a power-law decay without any intrinsic scale. Therefore, the pertinent scale is set by the edge of this inertial range—the outer scale. This results in long range correlations along the propagation direction, which contradicts the delta-correlation assumption. To some extent, the fact that the refractive index fluctuations are very small and thus allows light to propagate over long distances with minimal effect, mitigates this contradictory relationship between the step size and the outer scale. Still, our understanding of the evolution of photonic quantum states in turbulence would clearly benefit from a non-Markovian analysis.

In this paper, we employ a non-Markovian approach to study the evolution of photonic quantum states propagating through turbulence. We provide the derivation of a non-Markovian IPE, which takes the form of a second-order differential equation. The resulting equation has a form that does not in general have a solution. For this reason one needs to apply some simplifications or approximations to solve the differential equation. Here, we will show two such approaches. In the first approach we assume that the turbulence is weak, which allows one to perform a perturbative expansion of the solution for the differential equation. The weak turbulence conditions can be considered as complimentary to the SPS model, which implies strong turbulence conditions [17]. The second approach is to modify the functional form of the differential equation. The resulting differential equation then does have a solution. Here, we will only consider the single-photon case for the latter approach.

The paper is organized as follows. In section 2, we provide a brief review of background material, followed by a discussion of the approach that we will use to obtain a non-Markovian equation in section 3. The derivation of the non-Markovian IPE is shown in detail in section 4. We provide the two different approaches to find solutions for the non-Markovian IPE in sections 5 and 6, respectively. In section 7 we discuss some pertinent aspects of these analyses, followed by some conclusions in section 8.

2. Background

2.1. Notation

The discussions in this paper include both two-dimensional functions (such as the phase functions) and three-dimensional functions (such as the refractive index fluctuations). For this reason we need to define both two-dimensional and three-dimensional vectors to represent coordinate vectors. The two-dimensional coordinate vectors are always defined in the transverse plane, perpendicular to the propagation direction, the latter being the z-direction. For position coordinates, the two-dimensional position vector is denoted by a bold small $\mathbf{x}$, while the three-dimensional position vector is denoted by a bold capital $\mathbf{X}$. In the Fourier domain we prefer to work with spatial frequency vectors. The two-dimensional spatial frequency vector is denoted by a bold small $\mathbf{a}$, while the three-dimensional spatial frequency vector is denoted by a bold capital $\mathbf{A}$. Occasionally, we will also use the three-dimensional propagation vector, denoted by a bold capital $\mathbf{K} = 2\pi\mathbf{A}$. The small k is used to represent the wave number, which is not equal to $|\mathbf{K}|$.

During the analysis we will obtain expressions for density matrices in terms of different sets of coordinates. Instead of denoting all these density matrices by the same symbol ρ, we rather avoid possible confusion by using different symbols R, H, etc. to represent the density matrices, depending on their arguments. We only use ρ to represent the density matrix in generic discussions and for the final expressions of the solutions.

2.2. Scintillation

For a thin enough slab of the turbulent medium, one can represent the scintillation process as a phase modulation. The phase functions that represent the turbulent medium in such a modulation process are random functions taken from an ensemble of such functions. Each one is obtained from an element of the ensemble of refractive index fluctuations $\delta n(\mathbf{X})$, by an integration along the direction of propagation—the z-direction. The phase functions are therefore defined by

$$\theta(\mathbf{x}) = k\int_0^z \delta n(\mathbf{X})\, dz, \tag{1}$$

where k is the wavenumber, given as $k = 2\pi/\lambda$ in terms of the wavelength λ.

In the calculations of the evolution process, one often finds ensemble averages over phase functions, which give rise to the phase autocorrelation function. The latter is defined by

$$B_\theta(\Delta\mathbf{x}) = \mathcal{E}\{\theta(\mathbf{x}_1)\theta(\mathbf{x}_2)\}, \tag{2}$$

where $\mathcal{E}\{\cdot\}$ denotes the ensemble average and $\Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$. The autocorrelation function is also referred to as a covariance function, because these random functions are assumed to have zero mean.

A similar definition exists for the refractive index autocorrelation function

$$B_n(\Delta\mathbf{X}) = \mathcal{E}\{\delta n(\mathbf{X}_1)\delta n(\mathbf{X}_2)\}, \tag{3}$$

where $\Delta\mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2$. The refractive index autocorrelation function is related to the refractive index structure function in the following way

$$D_n(\Delta\mathbf{X}) = 2B_n(0) - 2B_n(\Delta\mathbf{X}). \tag{4}$$

In the Kolmogorov theory [18] the refractive index structure function is given by

$$D_n(\Delta\mathbf{X}) = C_n^2\, |\Delta\mathbf{X}|^{2/3}, \tag{5}$$

where C_n^2 is the refractive index structure constant.

The Wiener–Kinchine theorem [19] relates autocorrelation functions to power spectral density functions. For the refractive index autocorrelation function we have

$$B_n(\mathbf{X}) = \int \Phi_n(\mathbf{K})\exp(-i2\pi\mathbf{A}\cdot\mathbf{X})\, d^3a, \tag{6}$$

where $\Phi_n(\mathbf{K})$ is the refractive index power spectral density. In the Kolmogorov theory, the latter is given as [18]

$$\Phi_n(\mathbf{K}) = 0.033(2\pi)^3 C_n^2\, |\mathbf{K}|^{-11/3} \tag{7}$$

and the extra $(2\pi)^3$ factor is due to a difference in the definition of the Fourier transform [15]. The phase autocorrelation function is expressed as

$$B_\theta(\mathbf{x}) = \int \Phi_\theta(\mathbf{a})\exp(-i2\pi\mathbf{a}\cdot\mathbf{x})\, d^2a, \tag{8}$$

in terms of the phase power spectral density $\Phi_\theta(\mathbf{a})$.

Using equations (1)–(3), we express the two-dimensional phase autocorrelation function in terms of the three-dimensional refractive index autocorrelation function

$$B_\theta(\Delta\mathbf{x}) = k^2\int_0^z\int_0^z \mathcal{E}\{\delta n(\mathbf{X}_1)\delta n(\mathbf{X}_2)\}\,\mathrm{d}z_1\mathrm{d}z_2 = k^2\int_0^z\int_0^z B_n(\Delta\mathbf{X})\,\mathrm{d}z_1\,\mathrm{d}z_2. \quad (9)$$

Substituting equations (6) and (8) into (9), we obtain an expression for the phase power spectral density in terms of the refractive index power spectral density

$$\Phi_\theta(\mathbf{a}) = k^2\int\int_{z_0}^z\int_{z_0}^z \exp[-\mathrm{i}2\pi(z_1 - z_2)c] \times \Phi_n(\mathbf{K})\,\mathrm{d}z_2\,\mathrm{d}z_1\,\mathrm{d}c, \quad (10)$$

where c is related to the 'z-component' of $\mathbf{K}$ ($k_z = 2\pi c$). The integrals over z indicate that the refractive index fluctuations over the entire propagation path up to z contribute to the behavior at z.

2.3. Multiple phase screens

The infinitesimal propagation principle, which allows a MPS approach, follows from considering the change in the photonic state due to an infinitesimal propagation through the medium. The operation of such an infinitesimal propagation on the density operator can be expressed by

$$\hat{\rho}(z) \to \hat{\rho}(z+\delta z) = \mathrm{d}U\hat{\rho}(z)\mathrm{d}U^\dagger, \quad (11)$$

where $\mathrm{d}U$ is a unitary operator representing the infinitesimal propagation through the turbulent medium. When the density operator is expressed as a density matrix in terms of some arbitrary discrete basis $|m\rangle$, the output density matrix elements, after the infinitesimal propagation, are given by

$$\rho_{mn}(z+\delta z) = \sum_{pq}\langle m|\,\mathrm{d}U\,|p\rangle\rho_{pq}(z)\langle q|\,\mathrm{d}U^\dagger\,|n\rangle. \quad (12)$$

Using the paraxial wave equation in an inhomogeneous medium, given by [18]

$$\nabla_T^2\,g(\mathbf{X}) - \mathrm{i}2k\partial_z g(\mathbf{X}) + 2k^2\delta n(\mathbf{X})g(\mathbf{X}) = 0, \quad (13)$$

where $g(\mathbf{X})$ is the scalar electric field, one can show that [16]

$$\langle m|\,\mathrm{d}U\,|p\rangle = \delta_{mp} + \mathrm{i}\delta z\,\mathcal{P}_{mp} + \delta z\,\mathcal{L}_{mp}, \quad (14)$$

where δ_{mp} is the Kronecker delta,

$$\mathcal{P}_{mp}(z) \triangleq \frac{2\pi^2}{k}\int|\mathbf{a}|^2 G_m^*(\mathbf{a}, z)G_p(\mathbf{a}, z)\,\mathrm{d}^2a \quad (15)$$

and

$$\mathcal{L}_{mp}(z) \triangleq -\mathrm{i}k\iint G_m^*(\mathbf{a}, z)N(\mathbf{a}-\mathbf{a}', z)G_p(\mathbf{a}', z)\,\mathrm{d}^2a\,\mathrm{d}^2a'. \quad (16)$$

Here, $G_m(\mathbf{a}, z)$ and $N(\mathbf{a}, z)$ represent the transverse two-dimensional Fourier transforms of $g_m(\mathbf{X}) = \langle x|m\rangle$ and $\delta n(\mathbf{X})$, respectively.

The infinitesimal propagation of the density operator then leads to the following equation for each element in the ensemble [16]

$$\rho_{mn}(z_0+\delta z) = \rho_{mn}(z_0) + \mathrm{i}\delta z[\mathcal{P}, \rho(z_0)]_{mn} + \delta z\sum_p[\mathcal{L}_{mp}(z_0)\rho_{pn}(z_0) + \rho_{mp}(z_0)\mathcal{L}_{pn}^\dagger(z_0)]. \quad (17)$$

The right-hand side of equation (17) can be represented by an integral over a small range of z to replace the factor of δz. If one were to compute the ensemble average of equation (17), the dissipative term (sum over p) would vanish, because $\mathcal{E}\{N\} = 0$. One needs an expression with terms that are second-order in N before computing the ensemble averages to have non-zero dissipative terms. The result of such ensemble averages would then contain autocorrelation functions of $N(\mathbf{a}, z)$.

2.4. Markov approximation

The Markov approximation enters at the point where one computes the autocorrelation function of $N(\mathbf{a}, z)$

$$\Gamma(\mathbf{a}_1, \mathbf{a}_2, z_1, z_2) = \mathcal{E}\{N(\mathbf{a}_1, z_1)N^*(\mathbf{a}_2, z_2)\}. \quad (18)$$

One can model $N(\mathbf{a}, z)$ as

$$N(\mathbf{a}, z) = \int\left[\frac{\Phi_n(\mathbf{K})}{\Delta^3}\right]^{1/2}\tilde{\chi}(\mathbf{A})\exp(-\mathrm{i}2\pi cz)\,\mathrm{d}c, \quad (19)$$

where Δ is a correlation length in the frequency domain and $\tilde{\chi}(\mathbf{A})$ is a normally distributed, delta-correlated, random complex function, with a zero mean. Hence

$$\mathcal{E}\{\tilde{\chi}(\mathbf{A}_1)\tilde{\chi}^*(\mathbf{A}_2)\} = \Delta^3\delta_3(\mathbf{A}_1 - \mathbf{A}_2). \quad (20)$$

Since $\delta n(\mathbf{X})$ is a real-valued function, the random complex functions also have the property $\tilde{\chi}^*(\mathbf{A}) = \tilde{\chi}(-\mathbf{A})$. With the aid of equation (19), we write equation (18) as

$$\Gamma(\mathbf{a}_1, \mathbf{a}_2, z_1, z_2) = \delta_2(\mathbf{a}_1 - \mathbf{a}_2) \times \int \exp[-\mathrm{i}2\pi(z_1 - z_2)c_1]\Phi_n(\mathbf{K}_1)\,\mathrm{d}c_1. \quad (21)$$

In the Markov approximation, it is assumed that only the values of the field and the medium at z contribute to the behavior at z. This assumption implies that the refractive index fluctuations are delta-correlated along the z-direction. In the Fourier domain, the spectrum of the refractive index fluctuations would then be constant along the z-direction. In other words, the spectrum only depends on the transverse Fourier coordinates $\mathbf{a}$. The result is that one can substitute $k_z = 0$ ($c = 0$) in $\Phi_n(\mathbf{K})$. Making this substitution and evaluating the integrals in equation (10), one arrives at a simpler relationship given by

$$\Phi_\theta(\mathbf{a}) = zk^2\Phi_n(2\pi\mathbf{a}, 0). \quad (22)$$

The simpler expression for $\Phi_\theta(\mathbf{a})$ can in turn be used to simplify the model for N:

$$N(\mathbf{a}, z) = \tilde{\chi}(\mathbf{a})\left[\frac{\Phi_\theta(\mathbf{a})}{\Delta^2}\right]^{1/2}, \quad (23)$$

where $\tilde{\chi}(\mathbf{a})$ is now a two-dimensional random function, but other than that has the same properties as $\tilde{\chi}(\mathbf{A})$.

The Markov approximation is introduced into equation (21) by setting $k_z = 0$ in $\Phi_n(\mathbf{K})$, which gives

$$\Gamma(\mathbf{a}_1, \mathbf{a}_2, z_0, z) \approx \frac{\delta z}{2}\delta_2(\mathbf{a}_1 - \mathbf{a}_2)\Phi_n(2\pi\mathbf{a}_1, 0), \tag{24}$$

where $\delta z = z - z_0$. The factor of δz leads to a first-order differential equation—the Markovian IPE [16].

3. Non-Markovian approach

For the non-Markovian approach, we proceed without setting $k_z = 0$ in $\Phi_n(\mathbf{K})$. As a result, the integrals in equation (10) need to be evaluated by using an explicit expression for $\Phi_n(\mathbf{K})$. On the other hand, one can exploit the fact that δz is small for infinitesimal propagations. This allows one to expand equation (21) up to leading order in δz. As a result we have

$$\Gamma(\mathbf{a}_1, \mathbf{a}_2, z_0, z) \approx \frac{\delta z^2}{2}\delta_2(\mathbf{a}_1 - \mathbf{a}_2)\int \Phi_n(\mathbf{K}_1)\, \mathrm{d}c_1. \tag{25}$$

The factor of δz^2 (instead of just δz) suggests that the non-Markovian equation could be a second-order differential equation.

In the derivation in section 4 and appendix A, we will find that $z_1 = z_2 = z$. Thus, the correlation function in equation (18) or (21) becomes independent of z

$$\begin{aligned}\Gamma(\mathbf{a}_1, \mathbf{a}_2) &= \mathcal{E}\{N(\mathbf{a}_1, z)N^*(\mathbf{a}_2, z)\}\\ &= \delta_2(\mathbf{a}_1 - \mathbf{a}_2)\Phi_1(\mathbf{a}_1),\end{aligned} \tag{26}$$

where

$$\Phi_1(\mathbf{a}_1) \triangleq \int \Phi_n(\mathbf{K}_1)\, \mathrm{d}c_1. \tag{27}$$

Master equations for non-Markovian systems (for example, the Nakajima–Zwanzig equation [20, 21]) in general have the form

$$\partial_z\rho(z) = \int_{z_0}^{z} K(z, z')\rho(z')\, \mathrm{d}z', \tag{28}$$

where $K(z, z')$ is a super-operator that represents the memory in the system. Taking another derivative with respect to z on both sides

$$\partial_z^2\rho(z) = K(z, z)\rho(z) + \int_{z_0}^{z} [\partial_z K(z, z')]\rho(z')\, \mathrm{d}z', \tag{29}$$

one finds that the right-hand side still contains an integral over z. It is therefore not in general possible to describe non-Markovian systems in terms of a pure second-order differential equation, having no integrals over z. If, however, $K(z, z') = K(z')$ in equation (28), one would obtain

$$\partial_z^2\rho(z) = K(z)\rho(z). \tag{30}$$

In the particular case under consideration, it is possible to obtain a pure second-order differential equation, without integrals over z. Consider equation (17), expressed as a first-order differential equation

$$\begin{aligned}\partial_z\rho_{mn}(z) &= \mathrm{i}[\mathcal{P}(z), \rho(z)]_{mn}\\ &\quad + \sum_p[\mathcal{L}_{mp}(z)\rho_{pn}(z) + \rho_{mp}(z)\mathcal{L}_{pn}^{\dagger}(z)].\end{aligned} \tag{31}$$

If one differentiates equation (31) on both sides with respect to z, replaces the resulting first derivatives $\partial_z\rho(z)$ again by equation (31) and computes the ensemble average, by taking into account that $\mathcal{E}\{N\} = \mathcal{E}\{\partial_z N\} = 0$, one obtains a result without z-integrals that reads

$$\begin{aligned}\partial_z^2\rho_{mn}(z) &= \mathrm{i}[\partial_z\mathcal{P}(z), \rho(z)]_{mn} - [\mathcal{P}(z), [\mathcal{P}(z), \rho(z)]]_{mn}\\ &\quad + \sum_{p,q}\mathcal{E}\{2\mathcal{L}_{mp}(z)\rho_{pq}(z)\mathcal{L}_{qn}^{\dagger}(z)\\ &\quad - \mathcal{L}_{mp}^{\dagger}(z)\mathcal{L}_{pq}(z)\rho_{qn}(z)\\ &\quad - \rho_{mp}(z)\mathcal{L}_{pq}^{\dagger}(z)\mathcal{L}_{qn}(z)\}.\end{aligned} \tag{32}$$

Here we used the fact that the $\mathcal{L}$'s are anti-hermitian: $\mathcal{L}_{mn}^{\dagger} = -\mathcal{L}_{mn}$.

Using equations (16), (23) and (26), we compute the ensemble average over the $\mathcal{L}_{pq}$'s. The result is [16]

$$\begin{aligned}\Lambda_{mnpq} &\triangleq \mathcal{E}\{\mathcal{L}_{mp}(z)\mathcal{L}_{qn}^{\dagger}(z)\}\\ &= k^2\int W_{mp}(\mathbf{a}, z)W_{nq}^*(\mathbf{a}, z)\Phi_1(\mathbf{a})\, \mathrm{d}^2a,\end{aligned} \tag{33}$$

where

$$W_{ab}(\mathbf{a}, z) \triangleq \int G_a^*(\mathbf{a}' + \mathbf{a}, z)G_b(\mathbf{a}', z)\, \mathrm{d}^2a'. \tag{34}$$

When two of the indices on the $\mathcal{L}_{pq}$'s are contracted, one can use the orthogonality and completeness conditions of the modal basis to show that [16]

$$\sum_p\Lambda_{mnpp} = \delta_{mn}k^2\int\Phi_1(\mathbf{a})\, \mathrm{d}^2a \triangleq \delta_{mn}\Lambda_T. \tag{35}$$

Substituting equations (33) and (35) into equation (32), we obtain

$$\begin{aligned}\partial_z^2\rho_{mn}(z) &= \mathrm{i}[\partial_z\mathcal{P}(z), \rho(z)]_{mn} - [\mathcal{P}(z), [\mathcal{P}(z), \rho(z)]]_{mn}\\ &\quad + 2k^2\int\sum_{p,q}W_{mp}(\mathbf{a}, z)\rho_{pq}(z)W_{qn}^{\dagger}(\mathbf{a}, z)\\ &\quad \times \Phi_1(\mathbf{a})\, \mathrm{d}^2a - 2\Lambda_T\rho_{mn}(z).\end{aligned} \tag{36}$$

The result in equation (36) is a general expression for the non-Markovian IPE in an arbitrary discrete basis for a single photon propagating through turbulence. The commutators in equation (36) are the kinetic terms of the equation and represents the unitary evolution of the state. The second and third line in equation (36) contain the dissipative terms responsible for the decay in the coherence of the state.

Comparing equation (36) with the equivalent Markovian Lindblad equation (equation (62) in [16] with minor changes in notation)

$$\partial_z \rho_{mn}(z) = \mathrm{i}[\mathcal{P}(z), \rho(z)]_{mn} + \frac{k^2}{2}\int \Phi_0(\mathbf{a}) \sum_{p,q} [2W_{mp}(\mathbf{a}, z) \times \rho_{pq}(z) W^{\dagger}_{qn}(\mathbf{a}, z) - W^{\dagger}_{mp}(\mathbf{a}, z) W_{pq}(\mathbf{a}, z) \rho_{qn}(z) - \rho_{mp}(z) W^{\dagger}_{pq}(\mathbf{a}, z) W_{qn}(\mathbf{a}, z)]\, \mathrm{d}^2 a, \quad (37)$$

keeping in mind the definition of Λ_T given in equation (35), we note that the dissipative part of these equations are the same. Since the form of the Lindblad equation, such as the one shown in equation (37), ensures that the density matrix remains completely positive and preserves its trace, the same would therefore be true for equation (36).

Note, however, that the definitions of Φ_0 and Φ_1 are different in the two equations. In the Markovian case, given by equation (37), we have $\Phi_0(\mathbf{a}) = \Phi_n(2\pi\mathbf{a}, 0)$. Whereas for the non-Markovian case given in equation (36), Φ_1 is defined by an integration of Φ_n, as shown in equation (27). The difference in the definitions of Φ_0 and Φ_1, leads to a difference in their dimensions, as required by the fact that the number of derivative with respect the z is different in the two equations. For the Markovian case, $\Phi_0(\mathbf{a})$ has the dimension of distance cubed. In the context of the Lindblad equation, one can interpret $\Phi_0(\mathbf{a})$ as acting as (being proportional to) a decay constant γ. As such the rate of exponential decay of each of the respective Fourier components, labeled by $\mathbf{a}$, is governed by the value of $\Phi_1(\mathbf{a})$.

For the non-Markovian case, $\Phi_1(\mathbf{a})$ carries the dimension of distance squared. Since the non-Markovian master equation in (36) is a second-order differential equation, the individual Fourier components will undergo oscillations, instead of exponential decay. In this context, $\Phi_1(\mathbf{a})$ provides (is proportional to) the (spatial) frequency for the oscillations of each of the respective Fourier components, labeled by $\mathbf{a}$. The mechanism for the decay in the coherence of the state is in this case provided by the dephasing among all the different Fourier components due to their different spatial frequencies, given by the different function values of $\Phi_1(\mathbf{a})$. However, due to the oscillations, it may be possible to observe coherence revivals at particular propagation distances, depending on the nature of the input state.

Below, we will repeat this derivation in detail, but we will perform the derivation in the plane wave basis (Fourier domain), which is more beneficial for the purpose of finding solutions for the differential equation [17].

4. The non-Markovian IPE

In the transverse Fourier domain, the paraxial wave equation in an inhomogeneous medium is given by

$$\partial_z G(\mathbf{a}, z) = \mathrm{i}\pi\lambda\, |\mathbf{a}|^2 G(\mathbf{a}, z) - \mathrm{i}kN(\mathbf{a}, z) \star G(\mathbf{a}, z), \quad (38)$$

where $G(\mathbf{a}, z)$ is the transverse two-dimensional Fourier transforms of $g(\mathbf{X})$ and $\star$ denotes convolution. The first term on the right-hand side of equation (38) represents free-space propagation and the second term produces distortions due to the effect of the medium.

It is convenient to work in a 'rotating' frame in which the free-space term is removed. This is done by using

$$G(\mathbf{a}, z) = F(\mathbf{a}, z) \exp(\mathrm{i}\pi\lambda z\, |\mathbf{a}|^2), \quad (39)$$

to convert the paraxial wave equation in (38) into

$$\partial_z F(\mathbf{a}, z) = -\mathrm{i}k \int N(\mathbf{a} - \mathbf{u}, z) F(\mathbf{u}, z) \times \exp[-\mathrm{i}\pi\lambda z(|\mathbf{a}|^2 - |\mathbf{u}|^2)]\, \mathrm{d}^2 u. \quad (40)$$

To derive a non-Markovian IPE for a single-photon input state from equation (40), we assume that the input is a single-photon pure state in the plane wave basis, given (in the rotating frame) by

$$R(\mathbf{a}_1, \mathbf{a}_2, z) = F(\mathbf{a}_1, z) F^*(\mathbf{a}_2, z). \quad (41)$$

The details of the derivation of the non-Markovian IPE for the single-photon input state in equation (41) is shown in appendix A. The result is

$$\partial_z^2 R(\mathbf{a}_1, \mathbf{a}_2, z) = 2k^2 \int \{R(\mathbf{a}_1 - \mathbf{u}, \mathbf{a}_2 - \mathbf{u}, z) \exp[-\mathrm{i}2\pi\lambda z(\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{u}] - R(\mathbf{a}_1, \mathbf{a}_2, z)\} \Phi_1(\mathbf{u})\, \mathrm{d}^2 u. \quad (42)$$

Although it has an integral over the Fourier variables $\mathbf{u}$, the non-Markovian IPE is a second-order differential equation without any integrals over z.

The expression, equivalent to equation (42), for the two-photon states is given by

$$\partial_z^2 R(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z) = 2k^2 \int \{R(\mathbf{a}_1 - \mathbf{u}, \mathbf{a}_2 - \mathbf{u}, \mathbf{a}_3, \mathbf{a}_4, z) \times \exp[-\mathrm{i}2\pi\lambda z(\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{u}] + R(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 - \mathbf{u}, \mathbf{a}_4 - \mathbf{u}, z) \times \exp[-\mathrm{i}2\pi\lambda z(\mathbf{a}_3 - \mathbf{a}_4) \cdot \mathbf{u}] - 2R(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z)\} \Phi_1(\mathbf{u})\, \mathrm{d}^2 u. \quad (43)$$

To aid the solution of the non-Markovian IPE, we cast it in a form that decouples the z-dependence from the Fourier variables. This is done in a similar way as in [17], by performing the following two steps, converting $R \to S \to H$, all of which represent density matrices, but in terms of different sets of coordinates.

First, we redefine the Fourier variables (spatial frequencies) in terms of sums and differences, defined by

$$\mathbf{a}_1 = \mathbf{a}_s + \frac{1}{2}\mathbf{a}_d, \quad (44)$$

$$\mathbf{a}_2 = \mathbf{a}_s - \frac{1}{2}\mathbf{a}_d. \quad (45)$$

The state is then also refined

$$R(\mathbf{a}_1, \mathbf{a}_2, z) = R\left(\mathbf{a}_s + \frac{1}{2}\mathbf{a}_d, \mathbf{a}_s - \frac{1}{2}\mathbf{a}_d, z\right) \triangleq S(\mathbf{a}_s, \mathbf{a}_d, z). \quad (46)$$

The expression in (42) then becomes

$$\partial_z^2 S(\mathbf{a}_s, \mathbf{a}_d, z) = 2k^2 \int [S(\mathbf{a}_s - \mathbf{u}, \mathbf{a}_d, z)\exp(-i2\pi\lambda z\mathbf{a}_d \cdot \mathbf{u}) - S(\mathbf{a}_s, \mathbf{a}_d, z)]\Phi_1(\mathbf{u})\, d^2u. \quad (47)$$

The next step is to perform an inverse Fourier transform with respect to the sum coordinates:

$$H(\mathbf{x}, \mathbf{a}_d, z) = \int S(\mathbf{a}_s, \mathbf{a}_d, z)\exp(-i2\pi\mathbf{a}_s \cdot \mathbf{x})\, d^2a_s. \quad (48)$$

Equation (47) then reads

$$\partial_z^2 H(\mathbf{x}, \mathbf{a}_d, z) = -2k^2 Q(\lambda z\mathbf{a}_d + \mathbf{x})H(\mathbf{x}, \mathbf{a}_d, z), \quad (49)$$

where

$$Q(\mathbf{x}) \triangleq \int [1 - \exp(-i2\pi\mathbf{x} \cdot \mathbf{u})]\Phi_1(\mathbf{u})\, d^2u. \quad (50)$$

By combining the integral in equation (50) with the definition in equation (27), we find that $Q(\mathbf{x})$ is related to the refractive index structure function, with $z = 0$

$$Q(\mathbf{x}) = \frac{1}{2}D_n(\mathbf{x}, 0) = \frac{1}{2}C_n^2\, |\mathbf{x}|^{2/3}. \quad (51)$$

Here we used the refractive index structure function from Kolmogorov theory, given in equation (5). Note that this structure function is only valid within the inertial range between the inner and outer scales. The use of Kolmogorov theory may therefore lead to divergences, for which one would need to introduce some scale (typically the outer scale) to regularize the integrals. Using the expression in (51), we obtain an expression for the single-photon non-Markovian IPE, given by

$$\partial_z^2 H(z) = -k^2 C_n^2\, H(z)|\lambda z\mathbf{a}_d + \mathbf{x}|^{2/3}, \quad (52)$$

where $H(z) \equiv H(\mathbf{x}, \mathbf{a}_d, z)$. The equivalent expression for the two-photon case is

$$\partial_z^2 H(z) = -\, k^2 C_n^2\, H(z)(|\lambda z\mathbf{a}_{1d} + \mathbf{x}_1|^{2/3} + |\lambda z\mathbf{a}_{2d} + \mathbf{x}_2|^{2/3}), \quad (53)$$

where $H(z) \equiv H(\mathbf{x}_1, \mathbf{a}_{1d}, \mathbf{x}_2, \mathbf{a}_{2d}, z)$.

The second-order differential equations obtained in (52) and (53) are of the form given in equation (30), which does not have a general solution. To solve these differential equations, we use some approximations, discussed in the following two sections.

5. Perturbative solution

For the first method to solve the differential equation in (52), we assume that the turbulence is weak enough (C_n in equation (52) is small enough) to allow a perturbative approach. This approach has the benefit that it can be generalized to the two-photon case, but first we will consider the single-photon case.

5.1. Single-photon state

Consider a second-order differential equation having the form of equation (52) (or equation (30)), with the coupling constant g, which is proportional to the turbulence strength, made explicit

$$\partial_z^2 H(z) = gK(z)H(z). \quad (54)$$

Expand the solution as a Maclaurin series in g

$$H(z) = H_0(z) + gH_1(z) + g^2H_2(z) + \cdots \quad (55)$$

and substitute it back into equation (54). Setting $g = 0$, one obtains the zeroth-order perturbation

$$\partial_z^2 H_0(z) = 0. \quad (56)$$

Its solution must satisfy the initial conditions.

The two initial conditions for the second-order differential equation in (54) can be stated as follows:

(i) the initial rate of change of the state is zero

$$\partial_z H(z)|_{z=0} = 0 \quad (57)$$

and

(ii) the state at $z = 0$ is given by the input state

$$H(0) = H_{\text{in}}. \quad (58)$$

The solution of equation (56) that satisfies these initial conditions is

$$H_0(z) = H_{\text{in}}. \quad (59)$$

The first-order perturbation is obtained by taking a derivative with respect to g before setting $g = 0$. The resulting equation

$$\partial_z^2 H_1(z) = K(z)H_0(z) = K(z)H_{\text{in}}, \quad (60)$$

has a solution satisfying the initial conditions, given by

$$H_1(z) = H_{\text{in}} \int_0^z \int_0^{z_2} K(z_1)\, dz_1\, dz_2. \quad (61)$$

Here we will only go to sub-leading order in g. Therefore, our total solution, obtained from equations (59) and (61), is

$$H(z) = H_{\text{in}}\left[1 + \int_0^z \int_0^{z_2} K(z_1)\, dz_1\, dz_2\right], \quad (62)$$

where we reabsorbed g into $K(z)$. The validity of the solution in equation (62) is based on the assumption that the expansion parameter g is small. In appendix B we derive an expression for g in terms of the dimension parameters. Using this expression, one can determine the conditions under which the expansion parameter would be small enough for the perturbative solution to be valid.

To obtain an explicit expression for equation (62), one needs to evaluate the double z-integration of $K(z)$. The expression for $K(z)$ for the single-photon case, according to equation (52), is

$$K(z) = -k^2 C_n^2 (|\lambda z \mathbf{a}_d + \mathbf{x}|^2)^{1/3}. \tag{63}$$

The solution in equation (62) can thus be expressed as

$$H(z) = H_{\text{in}}\left[1 - k^2 C_n^2 \int_0^z \int_0^{z_2} P(z)^{1/3}\, dz_1\, dz_2\right], \tag{64}$$

where

$$P(z) = |\lambda z \mathbf{a}_d + \mathbf{x}|^2 = \lambda^2 z^2 |\mathbf{a}_d|^2 + 2\lambda z (\mathbf{a}_d \cdot \mathbf{x}) + |\mathbf{x}|^2. \tag{65}$$

The evaluation of the integrations over z in equation (64) is briefly discussed in appendix C. The result is given by

$$\begin{aligned} H(\mathbf{x}, \mathbf{a}_d, z) &= H_{\text{in}}(\mathbf{x}, \mathbf{a}_d)\Bigg\{1 \\ &+ k^2 C_n^2 \frac{(\mathbf{a}_d \cdot \mathbf{x})(|\mathbf{a}_d|^2 \lambda z + \mathbf{a}_d \cdot \mathbf{x})|\mathbf{a}_d \times \mathbf{x}|^{2/3}}{\lambda^2 |\mathbf{a}_d|^{14/3}} \\ &\times {}_2\mathrm{F}_1\left[\frac{-1}{3}, \frac{1}{2}; \frac{3}{2}; \frac{-(\mathbf{a}_d \cdot \mathbf{x})^2}{|\mathbf{a}_d \times \mathbf{x}|^2}\right] \\ &- k^2 C_n^2 \frac{(|\mathbf{a}_d|^2 \lambda z + \mathbf{a}_d \cdot \mathbf{x})^2 |\mathbf{a}_d \times \mathbf{x}|^{2/3}}{\lambda^2 |\mathbf{a}_d|^{14/3}} \\ &\times {}_2\mathrm{F}_1\left[\frac{-1}{3}, \frac{1}{2}; \frac{3}{2}; \frac{-(|\mathbf{a}_d|^2 \lambda z + \mathbf{a}_d \cdot \mathbf{x})^2}{|\mathbf{a}_d \times \mathbf{x}|^2}\right] + \frac{3}{8} k^2 C_n^2 \\ &\times \frac{[(|\mathbf{a}_d|^2 \lambda z + \mathbf{a}_d \cdot \mathbf{x})^2 + |\mathbf{a}_d \times \mathbf{x}|^2]^{4/3} - |\mathbf{a}_d|^{8/3} |\mathbf{x}|^{8/3}}{\lambda^2 |\mathbf{a}_d|^{14/3}}\Bigg\}, \end{aligned} \tag{66}$$

where $H_{\text{in}}(\mathbf{x}, \mathbf{a}_d)$ is the input state, ${}_2\mathrm{F}_1$ denotes a hypergeometrical function and where we used the identity

$$(\mathbf{A} \cdot \mathbf{B})^2 + |\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2 |\mathbf{B}|^2, \tag{67}$$

for abitrary vectors $\mathbf{A}$ and $\mathbf{B}$.

The expression in (66) has the form

$$H(\mathbf{x}, \mathbf{a}_d, z) = H_{\text{in}}(\mathbf{x}, \mathbf{a}_d) T(\mathbf{x}, \mathbf{a}_d, z), \tag{68}$$

where $T(\cdot)$ is the part in curly brackets in equation (66). It can be represented as

$$T(\mathbf{x}, \mathbf{a}_d, z) = 1 + gW(\mathbf{x}, \mathbf{a}_d, z). \tag{69}$$

Here $W(\mathbf{x}, \mathbf{a}_d, z)$, which is proportional to the part in equation (66) that is multiplied by C_n^2, represents the dissipative part of the kernel and

$$g \triangleq \frac{4\pi^4 C_n^2 w_0^{14/3}}{\lambda^4}, \tag{70}$$

is a dimensionless coupling constant. Its expression is obtained by rendering the expression in (64) in terms of dimensionless quantities, as shown in appendix B.

The expression in (66) depends on a mixture of Fourier and position domain coordinates. It is preferable to obtain an expression purely in terms of Fourier coordinates, to allow comparison with the previously obtained Markovian expression [17]. For this purpose we perform the steps of section 4 in reverse: $H \to S \to R$. Then we also multiply the result with the free-space propagation phase factor, converting $R \to \rho$. The resulting final expression is

$$\begin{aligned} \rho(\mathbf{a}_1, \mathbf{a}_2, z) &= \exp[\mathrm{i}\pi\lambda z(\mathbf{a}_1 - \mathbf{a}_2)^2] \int \rho_{\text{in}}(\mathbf{a}_1 - \mathbf{u}, \mathbf{a}_2 - \mathbf{u}, z) \\ &\times T(\mathbf{x}, \mathbf{a}_1 - \mathbf{a}_2, z) \exp(\mathrm{i}2\pi\mathbf{x} \cdot \mathbf{u})\, d^2x\, d^2u. \end{aligned} \tag{71}$$

Note that the integration over $\mathbf{x}$ will perform a Fourier transform of $T(\cdot)$ and the integral over $\mathbf{u}$ is a convolution of the input density matrix with the resulting kernel function.

In the absence of turbulence ($g = 0$), we have $T = 1$. The integral over $\mathbf{x}$ then gives a Dirac-delta function, which renders $\mathbf{u} = 0$ after integration over $\mathbf{u}$. The result gives the output density matrix as the input density matrix times the free-space propagation phase factor, as expected for free-space propagation without turbulence.

One can compare equation (71) with the Markovian expression (equation (22) in [17]), which has the form[1]

$$\begin{aligned} \rho(\mathbf{a}_1, \mathbf{a}_2, z) &= \exp[\mathrm{i}\pi\lambda z(\mathbf{a}_1 - \mathbf{a}_2)^2] \int \rho_{\text{in}}(\mathbf{a}_1 - \mathbf{u}, \mathbf{a}_2 - \mathbf{u}, z) \\ &\times K(\mathbf{u}, \mathbf{a}_1 - \mathbf{a}_2, z)\, d^2u, \end{aligned} \tag{72}$$

where $K(\cdot)$ is the Markovian kernel function, which corresponds to the Fourier transform of $T(\cdot)$. Hence, the final expression of the non-Markovian analysis has the same form as the previously obtained Markovian expression.

5.2. Two-photon state

The result in equation (71), together with equation (66), represents a perturbative solution for the single-photon differential equation given in equation (52). One can generalize this solution to the two-photon case. The general perturbative solution for the two-photon case, analogous to equation (62), is

$$\begin{aligned} H(z) = H_{\text{in}}\Bigg[1 &+ \int_0^z \int_0^{z_2} K_1(z_1)\, dz_1\, dz_2 \\ &+ \int_0^z \int_0^{z_2} K_2(z_1)\, dz_1\, dz_2\Bigg], \end{aligned} \tag{73}$$

where $K_1(z)$ and $K_2(z)$ are associated with the two photons, respectively.

In analogy to equation (69), the expression of $T(\cdot)$ that one would obtain for the two-photon case, after evaluating the integrals in equation (73), simply becomes

$$\begin{aligned} T_2(\mathbf{x}_1, \mathbf{a}_{1d}, \mathbf{x}_2, \mathbf{a}_{2d}, z) &= 1 \\ &+ gW(\mathbf{x}_1, \mathbf{a}_{1d}, z) + gW(\mathbf{x}_2, \mathbf{a}_{2d}, z), \end{aligned} \tag{74}$$

where $W(\cdot)$ is the same function as in equation (69).

6. Modified differential equation

The perturbative solution that we obtained above is strictly speaking only valid when the turbulence is fairly weak. It

[1] For the sake of the comparison we changed the sign of the integration variable $\mathbf{u}$ and expressed the normalized propagation distance t in terms of z.

would be useful to have another solution that is valid under more general turbulence conditions. Here we consider an alternative approach to solve the differential equation in (52). The idea is that, although the differential equation in (30) does not have a solution in general, it does have solutions when $K(z)$ has a particular functional form. In what follows, we will consider one such example.

The differential equation in (52) can be written as

$$\partial_z^2 H(z) = -k^2 C_n^2 P(z)^{1/3} H(z), \tag{75}$$

where $P(z)$ is given in equation (65). With the aid of equation (67), one can express $P(z)$ as

$$P(z) = \frac{[z\lambda\,|\mathbf{a}_d|^2 + (\mathbf{a}_d\cdot\mathbf{x})]^2 + |\mathbf{a}_d\times\mathbf{x}|^2}{|\mathbf{a}_d|^2}. \tag{76}$$

A special case that does allow a solution for equation (52), is when the cross-product term in equation (76) is neglected. The differential equation then has the form

$$\partial_z^2 H(z) = -\alpha^2 (z+\zeta)^{2/3} H(z), \tag{77}$$

where

$$\alpha = \frac{2\pi\,|\mathbf{a}_d|^{1/3}\sqrt{C_n^2}}{\lambda^{2/3}}, \tag{78}$$

$$\zeta = \frac{(\mathbf{a}_d\cdot\mathbf{x})}{\lambda\,|\mathbf{a}_d|^2}. \tag{79}$$

The modification that is applied to the differential equation assumes that the cross-product between these particular coordinate vectors gives a vanishing contribution to the final result. This assumption depends on the particular input optical field. For instance, if the input optical field is a Gaussian beam, then the expectation value of the cross-product between these coordinate vectors is zero.

The differential equation in (77) has the solution

$$H(z) = C_1\sqrt{z+\zeta}\; J_{3/8}\left[\frac{3\alpha}{4}(z+\zeta)^{4/3}\right] + C_2\sqrt{z+\zeta}\; Y_{3/8}\left[\frac{3\alpha}{4}(z+\zeta)^{4/3}\right], \tag{80}$$

where J_ν and Y_ν are Bessel functions of the first and second kind, respectively, and C_1 and C_2 are constant to be determined by the initial conditions, given in equations (57) and (58).

Applying the first initial condition given in equation (57), one finds that the constants must have the forms

$$C_1 = C_0 Y_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right), \tag{81}$$

$$C_2 = -C_0 J_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right), \tag{82}$$

where C_0 is a constant, common to both C_1 and C_2. Substituting equations (81) and (82) into (80), one obtains an interim expression for the solution, given by

$$H(z) = C_0\sqrt{z+\zeta}\left\{Y_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)J_{3/8}\left[\frac{3\alpha}{4}(z+\zeta)^{4/3}\right] - J_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)Y_{3/8}\left[\frac{3\alpha}{4}(z+\zeta)^{4/3}\right]\right\}. \tag{83}$$

Now we apply the second initial condition given in equation (58) to the expression in (83) to obtain

$$\begin{aligned} H(0) = H_{\text{in}} &= C_0\sqrt{\zeta}\left[Y_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)J_{3/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right) - J_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)Y_{3/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)\right] \\ &= \frac{8C_0}{3\pi\zeta^{5/6}\sqrt{\alpha}}, \end{aligned} \tag{84}$$

where we used the Wronskian

$$J_{\nu+1}(z)Y_\nu(z) - Y_{\nu+1}(z)J_\nu(z) = \frac{2}{\pi z}, \tag{85}$$

to obtain the last expression in (84). It gives a relationship between C_0 and H_{in}, which is then used to replace C_0 in equation (83). The resulting solution reads

$$\begin{aligned} H(z) = {} & \frac{3\pi}{8}H_{\text{in}}\zeta^{5/6}\sqrt{\alpha}\sqrt{z+\zeta} \\ & \times\left\{Y_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)J_{3/8}\left[\frac{3\alpha}{4}(z+\zeta)^{4/3}\right] - J_{-5/8}\left(\frac{3\alpha}{4}\zeta^{4/3}\right)Y_{3/8}\left[\frac{3\alpha}{4}(z+\zeta)^{4/3}\right]\right\}. \end{aligned} \tag{86}$$

The expression for the solution of the simplified differential equation that satisfies the initial conditions, is obtained from equation (86) by substituting equations (78) and (79), into it. We express the result as

$$\begin{aligned} H(\mathbf{x}, \mathbf{a}_d, z) = {} & \frac{\pi}{2}\beta H_{\text{in}}(\mathbf{x}, \mathbf{a}_d)(z\lambda\,|\mathbf{a}_d|^2 + \mathbf{a}_d\cdot\mathbf{x})^{1/2} \\ & \times\frac{(\mathbf{a}_d\cdot\mathbf{x})^{5/6}}{|\mathbf{a}_d|^{7/3}}\left\{Y_{-5/8}\left[\frac{\beta(\mathbf{a}_d\cdot\mathbf{x})^{4/3}}{|\mathbf{a}_d|^{7/3}}\right]\right. \\ & \times J_{3/8}\left[\frac{\beta(z\lambda\,|\mathbf{a}_d|^2 + \mathbf{a}_d\cdot\mathbf{x})^{4/3}}{|\mathbf{a}_d|^{7/3}}\right] \\ & - J_{-5/8}\left[\frac{\beta(\mathbf{a}_d\cdot\mathbf{x})^{4/3}}{|\mathbf{a}_d|^{7/3}}\right] \\ & \left.\times Y_{3/8}\left[\frac{\beta(z\lambda\,|\mathbf{a}_d|^2 + \mathbf{a}_d\cdot\mathbf{x})^{4/3}}{|\mathbf{a}_d|^{7/3}}\right]\right\}. \end{aligned} \tag{87}$$

Here $H_{\text{in}}(\mathbf{x}, \mathbf{a}_d)$ is the input state and

$$\beta \triangleq \frac{3\pi\sqrt{C_n^2}}{2\lambda^2} = \frac{3\sqrt{g}}{4\pi w_0^{7/3}}. \tag{88}$$

As with the solution of the perturbative analysis, the expression in (87) has the form given in equation (68), where the expression of $T(\cdot)$ is given by the right-hand side of equation (87), without $H_{\rm in}(\mathbf{x}, \mathbf{a}_d)$. In the Fourier domain, the solution therefore also takes the form given in equation (71), as one would expect.

Setting $\beta = 0$ ($g = 0$) in equation (87), one can show that $T = 1$. Therefore, in the absence of turbulence, one again recovers the normal free-space propagation in the same way as with the perturbative solution.

Unfortunately, the result given in equation (87) cannot be directly generalized to a two-photon case, as in the perturbative case above. The reason is that, when the simplification that was applied to equation (52) to give equation (77), is applied to equation (53), the resulting differential equation is not solvable.

7. Discussion

It is important to note that, although the system under investigation here deals with the evolution of a quantum state, it should not be confused with a non-Markovian quantum process. The latter concerns a situation where a system interacts with an environment such that the process needs to be described as an interacting quantum theory, formulated in terms of quantum mechanics. Such non-Markovian quantum processes are in general quite complex (see for instance [22]).

In contrast, the non-Markovianity that one encounters in the evolution of a photonic quantum state through turbulence is of a simpler nature. The process is essentially linear, having no interactions. One can see this from the fact that when light propagates through turbulent air, the effect of the light on the air is negligible (one does not see ripples in the air after the light passed through it). From a field theory perspective, one can say that the (scalar) field which represents the turbulent fluid has a large mass. Therefore, by assuming that this mass becomes infinite, one can 'integrate out' the scalar field, leaving a linear, non-interacting theory. Therefore, the current scenario does not have a quantum bath that acts as an environment and interacts with the system. In the case of light propagating through turbulence, the effect of the medium is simply a continuous modulation process with a fixed non-dynamical random function that extends over the propagation distance.

Here we only consider the MPS approach. Although the SPS model appears to follow from a Markovian approach, if one were to derive the SPS model using a non-Markovian approach, one would find that the leading contribution in the non-Markovian approach gives the same expression for the SPS model. The conclusions that one can derive from the SPS model are therefore applicable regardless of whether one considers a Markovian or non-Markovian approach.

One of the pertinent aspects of the SPS model is that it gives the behavior of the state in terms of a single dimensionless parameter $\mathcal{W} = w_0/r_0$, where w_0 is the optical beam waist radius and r_0 is the Fried parameter [23]. The relationship between the Rytov variance [18], which quantifies scintillation strength, and $\mathcal{W}$ indicates that, for a constant $\mathcal{W}$, the scintillation strength increases with propagation distance. Since, the SPS model is only valid under weak scintillation conditions, it breaks down when the propagation distance becomes too large. In the context of the evolution of an entangled quantum state propagating through turbulence, one finds that the SPS model can only describe this evolution correctly for the entire duration of a non-zero entanglement, if the turbulence is strong enough to complete this evolution over a relatively short propagation distance. As a result, one can conclude that the SPS model provides a tool to study quantum state evolution under strong turbulence conditions [17].

To compliment the SPS model one needs another model that can provide a tool to study quantum state evolution under weak turbulence conditions. For the Markovian approach, such a tool was presented in the form of the Markovian IPE [14, 17]. Here we provide such a tool for the non-Markovian approach, where we exploit the weakness of the turbulence to obtain a perturbative solution.

We also provide another solution for the non-Markovian IPE that does not assume weak turbulence. This is obtained by modifying the differential equation for the non-Markovian IPE. The resulting modified differential equation only works for the single-photon case. Its solution cannot be generalized to the two-photon case, because the simplification that is used does not render a readily solvable differential equation in the two-photon case. Nevertheless, it is not inconceivable that one may be able to find a simplification that can be applied to the two-photon differential equation which would allow solutions. The resulting expressions would in general be even more complex than those that we obtained here.

8. Conclusions

The propagation of a photonic quantum state through a turbulent atmosphere is considered in terms of a non-Markovian approach. This is done in contrast to the existing Markovian methods that have been proposed before. We derive a non-Markovian IPE, which takes the form of a second-order differential equation with respect to the propagation distance. The non-Markovian IPE contains no integrations over the propagation distance. The form of this second-order differential equation does not allow immediate solutions.

To solve the non-Markovian IPE, we follow two different approaches. The first is to assume the turbulence is weak enough to allow a perturbative analysis. This approach gives a solution that contains hyper-geometrical functions. Although we obtain the solution for the single-photon case, it can be generalized to the two-photon case.

The second approach is to apply a particular simplification to the form of the differential equation. The resulting simplified differential equation can be solved to give a solution in terms of Bessel functions of fractional order. It only applies to the single-photon case.

Appendix A. Derivation of the single-photon non-Markovian IPE

Here we show in detail the derivation of the single-photon non-Markovian IPE.

Differentiating equation (41) with respect to z, one obtains

$$\partial_z R(\mathbf{a}_1, \mathbf{a}_2, z) = [\partial_z F(\mathbf{a}_1, z)] F^*(\mathbf{a}_2, z) + F(\mathbf{a}_1, z)[\partial_z F^*(\mathbf{a}_2, z)]. \quad (A.1)$$

Substitution of equation (40) into (A.1) then leads to

$$\begin{aligned}\partial_z R(\mathbf{a}_1, \mathbf{a}_2, z) = & -\mathrm{i}k \int N(\mathbf{a}_1 - \mathbf{u}, z) F(\mathbf{u}, z) F^*(\mathbf{a}_2, z) \\ & \times \exp[-\mathrm{i}\pi\lambda z(|\mathbf{a}_1|^2 - |\mathbf{u}|^2)] \mathrm{d}^2 u \\ & + \mathrm{i}k \int N^*(\mathbf{a}_2 - \mathbf{u}, z) F(\mathbf{a}_1, z) F^*(\mathbf{u}, z) \\ & \times \exp[\mathrm{i}\pi\lambda z(|\mathbf{a}_2|^2 - |\mathbf{u}|^2)] \mathrm{d}^2 u.\end{aligned} \quad (A.2)$$

A second derivative with respect to z produces several terms on the right-hand side, but only those terms that contain derivatives of F and F^* will lead to terms that are second-order in N. All the other terms fall away when ensemble averaging is performed. Hence, retaining only those terms that will survive ensemble averaging, we obtain

$$\begin{aligned}\partial_z^2 R(\mathbf{a}_1, \mathbf{a}_2, z) = & -\mathrm{i}k \int N(\mathbf{a}_1 - \mathbf{u}, z)[F^*(\mathbf{a}_2, z)\partial_z F(\mathbf{u}, z) \\ & + F(\mathbf{u}, z)\partial_z F^*(\mathbf{a}_2, z)] \\ & \times \exp[-\mathrm{i}\pi\lambda z(|\mathbf{a}_1|^2 - |\mathbf{u}|^2)] \\ & + N^*(\mathbf{a}_2 - \mathbf{u}, z)[F^*(\mathbf{u}, z)\partial_z F(\mathbf{a}_1, z) \\ & + F(\mathbf{a}_1, z)\partial_z F^*(\mathbf{u}, z)] \\ & \times \exp[\mathrm{i}\pi\lambda z(|\mathbf{a}_2|^2 - |\mathbf{u}|^2)] \mathrm{d}^2 u.\end{aligned} \quad (A.3)$$

After substituting equation (40) and its complex conjugate into equation (A.3) for the second time, we have

$$\begin{aligned}\partial_z^2 R(\mathbf{a}_1, \mathbf{a}_2, z) = & k^2 \int \{2N(\mathbf{a}_1 - \mathbf{u}, z) \\ & \times N^*(\mathbf{a}_2 - \mathbf{v}, z) F(\mathbf{u}, z) F^*(\mathbf{v}, z) \\ & \times \exp[\mathrm{i}\pi\lambda z(|\mathbf{a}_2|^2 - |\mathbf{v}|^2 - |\mathbf{a}_1|^2 + |\mathbf{u}|^2)] \\ & - N(\mathbf{a}_1 - \mathbf{u}, z) N(\mathbf{u} - \mathbf{v}, z) F(\mathbf{v}, z) F^*(\mathbf{a}_2, z) \\ & \times \exp[-\mathrm{i}\pi\lambda z(|\mathbf{a}_1|^2 - |\mathbf{v}|^2)] \\ & - N^*(\mathbf{a}_2 - \mathbf{u}, z) N^*(\mathbf{u} - \mathbf{v}, z) F(\mathbf{a}_1, z) F^*(\mathbf{v}, z) \\ & \times \exp[\mathrm{i}\pi\lambda z(|\mathbf{a}_2|^2 - |\mathbf{v}|^2)]\} \, \mathrm{d}^2 u \, \mathrm{d}^2 v.\end{aligned} \quad (A.4)$$

Next we evaluate the ensemble average of equation (A.4), using equation (26), to obtain

$$\begin{aligned}\partial_z^2 R(\mathbf{a}_1, \mathbf{a}_2, z) = & k^2 \int \{2\delta_2(\mathbf{a}_1 - \mathbf{u} - \mathbf{a}_2 + \mathbf{v}) \\ & \times \Phi_1(\mathbf{a}_1 - \mathbf{u}) F(\mathbf{u}, z) F^*(\mathbf{v}, z) \\ & \times \exp[\mathrm{i}\pi\lambda z(|\mathbf{a}_2|^2 - |\mathbf{v}|^2 - |\mathbf{a}_1|^2 + |\mathbf{u}|^2)] \\ & - \delta_2(\mathbf{a}_1 - \mathbf{v}) \Phi_1(\mathbf{a}_1 - \mathbf{u}) F(\mathbf{v}, z) F^*(\mathbf{a}_2, z) \\ & \times \exp[-\mathrm{i}\pi\lambda z(|\mathbf{a}_1|^2 - |\mathbf{v}|^2)] \\ & - \delta_2(\mathbf{a}_2 - \mathbf{v}) \Phi_1(\mathbf{a}_2 - \mathbf{u}) F(\mathbf{a}_1, z) F^*(\mathbf{v}, z) \\ & \times \exp[\mathrm{i}\pi\lambda z(|\mathbf{a}_2|^2 - |\mathbf{v}|^2)]\} \, \mathrm{d}^2 u \, \mathrm{d}^2 v,\end{aligned} \quad (A.5)$$

where $\Phi_1(\mathbf{a}_1)$ is defined in equation (27). We evaluate the integrals over $\mathbf{v}$ to remove the Dirac-delta functions. After some simplification, one then obtains the expression for the single-photon non-Markovian IPE given in equation (42).

Appendix B. Coupling constant

To find an expression for a dimensionless coupling constant we express equation (62), together with equation (65), in terms of dimensionless coordinates and parameters. These are defined by normalizing the original coordinates with the aid of the characteristic dimension parameters

$$\begin{aligned}\mathbf{f} & \triangleq \pi w_0 \mathbf{a}_d, \\ \mathbf{r} & \triangleq \frac{\mathbf{x}}{w_0}, \\ t & \triangleq \frac{\lambda z}{\pi w_0^2}.\end{aligned} \quad (B.1)$$

Here, w_0 is the radius of the optical beam. In terms of these coordinates, the solution in equation (62) can be expressed as

$$H(t) = H_{\mathrm{in}}\left[1 - \frac{4\mathcal{T}}{\Theta^4}\int_0^t \int_0^{t_2} (|t\mathbf{f} + \mathbf{r}|^2)^{1/3} \, \mathrm{d}t_1 \, \mathrm{d}t_2\right], \quad (B.2)$$

where

$$\mathcal{T} \triangleq C_n^2 w_0^{2/3}, \quad (B.3)$$

is a normalized turbulence strength, and

$$\Theta \triangleq \frac{\lambda}{\pi w_0}, \quad (B.4)$$

is the Gaussian beam divergence angle. The dimensionless combination of the dimension parameters in front of the dissipative term gives us the expression for the effective coupling constant:

$$g \triangleq \frac{4\mathcal{T}}{\Theta^4} = \frac{4\pi^4 C_n^2 w_0^{14/3}}{\lambda^4}. \quad (B.5)$$

Appendix C. Integration of the structure function

The z-integrals in equation (64) can be expressed by

$$\mathcal{S} = \int_0^z \int_0^{z_2} (|\lambda z \mathbf{a}_d + \mathbf{x}|^2)^{1/3} \, \mathrm{d}z_1 \, \mathrm{d}z_2. \quad (C.1)$$

One can evaluate this integral in different ways, leading to expressions that may appear different, but represent the same function. Here, we only show one such approach, where we use a Dirac-delta function to remove the quadratic polynomial from under the power of $1/3$. The expression becomes

$$\mathcal{S} = \iint \int_0^z \int_0^{z_2} q_0^{1/3} \exp[\mathrm{i}2\pi b_0(q_0 - |\lambda z \mathbf{a}_d + \mathbf{x}|^2)] \times \mathrm{d}z_1 \, \mathrm{d}z_2 \, \mathrm{d}b_0 \, \mathrm{d}q_0. \quad (C.2)$$

An integration over b_0 will turn the exponential function into a Dirac-delta function.

First, we evaluate the z-integrations, which leads to an expression that contains error-functions. The error-functions are replaced by auxiliary integrals

$$\mathrm{erf}(A) \to \frac{2A}{\sqrt{\pi}} \int_0^1 \exp(-\xi^2 A^2)\, \mathrm{d}\xi. \tag{C.3}$$

Considering the b_0-integrals of the resulting expression, one finds two types of integrals. One is of the form that would produce Dirac-delta functions which are then removed after the q_0-integration. The other is of the form

$$\int \frac{\sin[(q_0 - U)b_0]}{b_0}\, \mathrm{d}b_0 = \mathrm{sign}(q_0 - U)\pi. \tag{C.4}$$

The sign-function separates the integration range of q_0 into two regions that add with opposite signs. Once both the b_0- and q_0-integrations are evaluated, one obtains

$$\begin{aligned} \mathcal{S} = &- \frac{3}{8} \frac{[|\mathbf{x}|^2 + 2\lambda z(\mathbf{a}_d \cdot \mathbf{x}) + \lambda^2 z^2 |\mathbf{a}_d|^2]^{4/3} - |\mathbf{x}|^{8/3}}{\lambda^2 |\mathbf{a}_d|^2} \\ &+ \frac{(\lambda z |\mathbf{a}_d|^2 + (\mathbf{a}_d \cdot \mathbf{x}))^2}{\lambda^2 |\mathbf{a}_d|^{14/3}} \\ &\times \int_0^1 [|\mathbf{a}_d \times \mathbf{x}|^2 + (|\mathbf{a}_d|^2 \lambda z + \mathbf{a}_d \cdot \mathbf{x})^2 \xi^2]^{1/3} \mathrm{d}\xi \\ &- \frac{(\mathbf{a}_d \cdot \mathbf{x})[\lambda z |\mathbf{a}_d|^2 + (\mathbf{a}_d \cdot \mathbf{x})]}{\lambda^2 |\mathbf{a}_d|^{14/3}} \\ &\times \int_0^1 [|\mathbf{a}_d \times \mathbf{x}|^2 + (\mathbf{a}_d \cdot \mathbf{x})^2 \xi^2]^{1/3} \mathrm{d}\xi. \end{aligned} \tag{C.5}$$

The remaining ξ-integrals are of the form

$$\int_0^1 (A^2 + B^2 \xi^2)^{1/3} \mathrm{d}\xi = (A^2)^{1/3}\ {}_2\mathrm{F}_1\left(\frac{-1}{3}, \frac{1}{2}; \frac{3}{2}; \frac{-B^2}{A^2}\right), \tag{C.6}$$

where ${}_2\mathrm{F}_1$ denotes a hyper-geometrical function. After evaluating the ξ-integrals and replacing the result into equation (64), we obtain equation (66).

References

[1] Pors B-J, Monken C H, Eliel E R and Woerdman J P 2011 Transport of orbital–angular–momentum entanglement through a turbulent atmosphere *Opt. Express* **19** 6671

[2] Hamadou Ibrahim A, Roux F S, McLaren M, Konrad T and Forbes A 2013 Orbital–angular–momentum entanglement in turbulence *Phys. Rev.* A **88** 012312

[3] Rodenburg B, Mirhosseini M, Malik M, Magaña-Loaiza O S, Yanakas M, Maher L, Steinhoff N K, Tyler G A and Boyd R W 2014 Simulating thick atmospheric turbulence in the lab with application to orbital angular momentum communication *New J. Phys.* **16** 033020

[4] Fedrizzi A, Ursin R, Herbst T, Nespoli M, Prevedel R, Scheidl T, Tiefenbacher F, Jennewein T and Zeilinger A 2009 High-fidelity transmission of entanglement over a high-loss free-space channel *Nat. Phys.* **5** 389–92

[5] Vallone G, D'Ambrosio V, Sponselli A, Slussarenko S, Marrucci L, Sciarrino F and Villoresi P 2014 Free-space quantum key distribution by rotation-invariant twisted photons *Phys. Rev. Lett.* **113** 060503

[6] Krenn M, Fickler R, Fink M, Handsteiner J, Malik M, Scheidl T, Ursin R and Zeilinger A 2014 Communication with spatially modulated light through turbulent air across Vienna *New J. Phys.* **16** 113028

[7] Paterson C 2005 Atmospheric turbulence and orbital angular momentum of single photons for optical communication *Phys. Rev. Lett.* **94** 153901

[8] Smith B J and Raymer M G 2006 Two-photon wave mechanics *Phys. Rev.* A **74** 062104

[9] Gopaul C and Andrews R 2007 The effect of atmospheric turbulence on entangled orbital angular momentum states *New J. Phys.* **9** 94

[10] Jha A K, Tyler G A and Boyd R W 2010 Effects of atmospheric turbulence on the entanglement of spatial two-qubit states *Phys. Rev.* A **81** 053832

[11] Malik M, O'Sullivan M, Rodenburg B, Mirhosseini M, Leach J, Lavery M P J, Padgett M J and Boyd R W 2012 Influence of atmospheric turbulence on optical communications using orbital angular momentum for encoding *Opt. Express* **20** 13195

[12] Gonzalez Alonso J R and Brun T A 2013 Protecting orbital–angular–momentum photons from decoherence in a turbulent atmosphere *Phys. Rev.* A **88** 022326

[13] Leonhard N D, Shatokhin V N and Buchleitner A 2015 Universal entanglement decay of photonic-orbital–angular–momentum qubit states in atmospheric turbulence *Phys. Rev.* A **91** 012345

[14] Roux F S 2011 Infinitesimal-propagation equation for decoherence of an orbital–angular–momentum-entangled biphoton state in atmospheric turbulence *Phys. Rev.* A **83** 053822

[15] Roux F S 2011 Infinitesimal-propagation equation for decoherence of an orbital–angular–momentum-entangled biphoton state in atmospheric turbulence *Phys. Rev.* A **83** 053822
Roux F S 2013 *Phys. Rev.* A **88** 049906 erratum

[16] Roux F S 2014 The Lindblad equation for the decay of entanglement due to atmospheric scintillation *J. Phys. A: Math. Theor.* **47** 195302

[17] Roux F S, Wellens T and Shatokhin V N 2015 Entanglement evolution of twisted photons in strong atmospheric turbulence *Phys. Rev.* A **92** 012326

[18] Andrews L C and Phillips R L 1998 *Laser Beam Propagation Through Random Media* (Washington: SPIE)

[19] Goodman J W 1985 *Statistical Optics* (New York: Wiley-Interscience)

[20] Nakajima S 1958 On quantum theory of transport phenomena steady diffusion *Prog. Theor. Phys.* **20** 948–59

[21] Zwanzig R 1960 Ensemble method in the theory of irreversibility *J. Chem. Phys.* **33** 1338–41

[22] Breuer H P and Petruccione F 2002 *The Theory of Open Quantum Systems* (New York: Oxford University Press)

[23] Fried D L 1966 Optical resolution through a randomly inhomogeneous medium for very long and very short exposures *J. Opt. Soc. Am.* **56** 1372–9